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ORIGINAL PAPER

Steady-state concentrations of carbon dioxide absorbed into phenyl glycidyl ether solutions by the Adomian decomposition method

Jun-Sheng Duan · Randolph Rach · Abdul-Majid Wazwaz

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Abstract In this paper, we examine a system of two coupled nonlinear differential equations that relates the concentrations of carbon dioxide CO_2 and phenyl glycidyl ether in solution. This system is subject to a set of Dirichlet boundary conditions and a mixed set of Neumann and Dirichlet boundary conditions. We apply the Adomian decomposition method combined with the Duan–Rach modified recursion scheme to analytically treat this system of coupled nonlinear boundary value problems. The rapid convergence of our analytic approximate solutions is demonstrated by graphs of the objective error analysis instead of comparison to an alternate solution technique alone. The Adomian decomposition method yields a rapidly convergent, easily computable, and readily verifiable sequence of analytic approximate solutions that is suitable for numerical parametric simulations. Thus our sequence of approximate solutions are shown to identically satisfy the original set of model equations as closely as we please.

Keywords Carbon dioxide · Phenyl glycidyl ether · Adomian decomposition method · Adomian polynomials

Mathematics Subject Classification 34B15 · 35A20 · 45D05

J.-S. Duan (✉)
School of Sciences, Shanghai Institute of Technology,
Shanghai 201418, People's Republic of China
e-mail: duanjssdu@sina.com; duanjs@sit.edu.cn

R. Rach
316 S. Maple St., Hartford, MI 49057-1225, USA
e-mail: tapstrike@gmail.com

A.-M. Wazwaz
Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
e-mail: wazwaz@sxu.edu

1 Introduction

Carbon dioxide is crucial in plant photosynthesis, the manufacturing of carbonated soft drinks, the powering of pneumatic systems in robots, used in fire extinguishers, removing caffeine from coffee, etc. [11,17,18]. Carbon dioxide is a useful gas that is composed of one carbon atom and two oxygen atoms [17]. Recently, the chemical fixation of carbon dioxide has become an important research topic, because of the danger posed by global warming and that the conversion of carbon dioxide into valuable substances is an extremely attractive solution [11,17,18].

The kinetics of the reaction between CO_2 and phenyl glycidyl ether (PGE) in solution has attracted much interest. Park et al. [18] and Choe et al. [10] have investigated the chemical absorption of carbon dioxide into PGE solutions containing the catalyst THA–CP–MS41 in a heterogeneous system. In [17], a cumbersome analytic expression needlessly involving logarithms for the steady-state concentrations of CO_2, PGE and the flux was determined by using a form of the earlier technique of the Adomian–Rach modified recursion scheme in the Adomian decomposition method formerly known as the double decomposition method [5,7,8].

The system of nonlinear differential equations that relates the steady-state concentrations of CO_2 and PGE was previously established [17] as

$$\frac{d^2u(x)}{dx^2} = \frac{\alpha_1 u(x)v(x)}{1+\beta_1 u(x)+\beta_2 v(x)}, \tag{1}$$

$$\frac{d^2v(x)}{dx^2} = \frac{\alpha_2 u(x)v(x)}{1+\beta_1 u(x)+\beta_2 v(x)}, \tag{2}$$

subject to the set of Dirichlet boundary conditions

$$u(0) = 1,\ u(1) = k, \tag{3}$$

and the mixed set of Neumann and Dirichlet boundary conditions

$$v'(0) = 0,\ v(1) = 1, \tag{4}$$

where the functions $u(x)$ and $v(x)$ are the concentrations of CO_2 and PGE, respectively, α_1, α_2, β_1 and β_2 are normalized system parameters, x is the dimensionless distance as measured from the center, and k is the dimensionless concentration of CO_2 at the surface of the catalyst [17].

In this work, we aim to apply the Adomian decomposition method [3–6,20,23] combined with the Duan–Rach modified recursion scheme [15,16,21] to systematically obtain a rapidly convergent analytic approximate solution that is convenient for numerical simulations. Furthermore, our approach is readily extensible to far more complicated systems without further ado. The rapid rate of convergence of our approximate solutions is validated by graphs of the error analysis that feature the error remainder functions and the maximal error remainder parameters instead of comparison to an alternate solution technique alone.

2 The Duan–Rach modified recursion scheme in the Adomian decomposition method

We rewrite Eqs. (1) and (2) in Adomian's operator-theoretic form as

$$Lu = N_1 u, v, \tag{5}$$
$$Lv = N_2 u, v, \tag{6}$$

where $N_j\, u, v = \alpha_j f\,(u\,(x)\,, v\,(x))$ for $j = 1, 2$, and the linear differential operator L and the composite nonlinearity are

$$L\,(\cdot) = \frac{d^2}{dx^2}\,(\cdot)\,, \quad f\,(u\,(x)\,, v\,(x)) = \frac{u\,(x)\,v\,(x)}{1 + \beta_1 u\,(x) + \beta_2 v\,(x)}. \tag{7}$$

Applying the corresponding inverse linear operator

$$L^{-1}\,(\cdot) = \int_0^x \int_0^x (\cdot)\, dx\, dx$$

to both sides of Eqs. (5) and (6) leads to

$$u\,(x) = u\,(0) + x\,u'\,(0) + L^{-1} N_1 u, v, \tag{8}$$
$$v\,(x) = v\,(0) + x\,v'\,(0) + L^{-1} N_2 u, v. \tag{9}$$

Substituting the known boundary values $u\,(0) = 1,\ v'\,(0) = 0$, we obtain the system of coupled nonlinear Volterra integral equations with two—as yet undetermined—constants of integration $u'\,(0)$ and $v\,(0)$ that constitutes an intermediate step as

$$u\,(x) = 1 + x\,u'\,(0) + L^{-1} N_1 u, v, \tag{10}$$
$$v\,(x) = v\,(0) + L^{-1} N_2 u, v. \tag{11}$$

For convenience in subsequent calculations, we define the definite integral operator as

$$L_1^{-1}\,(\cdot) = \int_0^1 \int_0^x (\cdot)\, dx\, dx.$$

Next we evaluate the two concentrations at the surface $x = 1$ using the remaining boundary values

$$u\,(1) = k, \quad v\,(1) = 1,$$

and Eqs. (10) and (11) to determine the values by formula of the remaining unknown constants of integration as

$$u'(0) = k - 1 - L_1^{-1} N_1 u, v, \tag{12}$$

$$v(0) = 1 - L_1^{-1} N_2 u, v. \tag{13}$$

Upon substitution of Eqs. (12) and (13) into Eqs. (10) and (11), we have incorporated all four boundary conditions to obtain the equivalent system of coupled nonlinear Fredholm–Volterra integral equations without any undetermined constants of integration

$$u(x) = 1 + (k-1)x - x L_1^{-1} N_1 u, v + L^{-1} N_1 u, v, \tag{14}$$

$$v(x) = 1 - L_1^{-1} N_2 u, v + L^{-1} N_2 u, v, \tag{15}$$

or, equivalently,

$$u(x) = 1 + (k-1)x - \alpha_1 x L_1^{-1} f(u(x), v(x)) + \alpha_1 L^{-1} f(u(x), v(x)), \tag{16}$$

$$v(x) = 1 - \alpha_2 L_1^{-1} f(u(x), v(x)) + \alpha_2 L^{-1} f(u(x), v(x)). \tag{17}$$

Applying the Adomian decomposition series, we decompose the two coupled solutions

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{18}$$

and the nonlinearity as the series of the two-variable Adomian polynomials

$$f(u(x), v(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{19}$$

where the two-variable Adomian polynomials are defined by the formula

$$\begin{aligned} A_n &= A_n(u_0, u_1, \ldots, u_n; v_0, v_1, \ldots, v_n) \\ &= \frac{1}{n!} \frac{d^n}{d\lambda^n} f\left(\sum_{j=0}^{n} u_j \lambda^j, \sum_{j=0}^{n} v_j \lambda^j \right) \Bigg|_{\lambda=0}. \end{aligned} \tag{20}$$

Other algorithms for the one-variable and multivariable Adomian polynomials have been proposed such as in [1,2,6,9,12–14,19,20,22,23]. Duan [12–14] has recently crafted several new, more efficient algorithms for fast generation of the one-variable and multivariable Adomian polynomials. For convenience, we list the first five two-variable Adomian polynomials of the general bivariate function $f(u, v)$ with the decompositions $u = \sum_{n=0}^{\infty} u_n$, $v = \sum_{n=0}^{\infty} v_n$ as follows,

$$A_0 = f(u_0, v_0),$$
$$A_1 = v_1 f^{(0,1)}(u_0, v_0) + u_1 f^{(1,0)}(u_0, v_0),$$

$$A_2 = v_2 f^{(0,1)} + \frac{1}{2} v_1^2 f^{(0,2)} + u_2 f^{(1,0)} + u_1 v_1 f^{(1,1)} + \frac{1}{2} u_1^2 f^{(2,0)},$$

$$A_3 = v_3 f^{(0,1)} + v_1 v_2 f^{(0,2)} + \frac{1}{6} v_1^3 f^{(0,3)} + u_3 f^{(1,0)} + (u_2 v_1 + u_1 v_2) f^{(1,1)} + \frac{1}{2} u_1 v_1^2 f^{(1,2)} + u_1 u_2 f^{(2,0)} + \frac{1}{2} u_1^2 v_1 f^{(2,1)} + \frac{1}{6} u_1^3 f^{(3,0)},$$

$$A_4 = v_4 f^{(0,1)} + \left(\frac{v_2^2}{2} + v_1 v_3 \right) f^{(0,2)} + \frac{1}{2} v_1^2 v_2 f^{(0,3)} + \frac{1}{24} v_1^4 f^{(0,4)} + u_4 f^{(1,0)} + (u_3 v_1 + u_2 v_2 + u_1 v_3) f^{(1,1)} + \left(\frac{1}{2} u_2 v_1^2 + u_1 v_1 v_2 \right) f^{(1,2)} + \frac{1}{6} u_1 v_1^3 f^{(1,3)} + \left(\frac{u_2^2}{2} + u_1 u_3 \right) f^{(2,0)} + \left(u_1 u_2 v_1 + \frac{1}{2} u_1^2 v_2 \right) f^{(2,1)} + \frac{1}{4} u_1^2 v_1^2 f^{(2,2)} + \frac{1}{2} u_1^2 u_2 f^{(3,0)} + \frac{1}{6} u_1^3 v_1 f^{(3,1)} + \frac{1}{24} u_1^4 f^{(4,0)},$$

where we use the notation $f^{(m,n)} = f^{(m,n)}(u_0, v_0) = \frac{\partial^{m+n} f}{\partial u^m \partial v^n}(u_0, v_0)$ as a space-saving shorthand.

For example, we display the first two Adomian polynomials tailored to the particular nonlinearity in (7) as

$$A_0 = \frac{u_0(x) v_0(x)}{\beta_1 u_0(x) + \beta_2 v_0(x) + 1}, \tag{21}$$

$$A_1 = \frac{u_1(x) v_0(x) (\beta_2 v_0(x) + 1) + u_0(x) v_1(x) (\beta_1 u_0(x) + 1)}{(\beta_1 u_0(x) + \beta_2 v_0(x) + 1)^2}. \tag{22}$$

MATHEMATICA code generating the two-variable Adomian polynomials of a general abstract function $f(u, v)$ based on the algorithm in Theorem 1 [14] is listed in "Appendix 1".

Upon substitution of the decomposition series (18) and (19) into Eqs. (16) and (17), we obtain

$$\sum_{n=0}^{\infty} u_n(x) = 1 + (k-1)x - \alpha_1 x L_1^{-1} \sum_{n=0}^{\infty} A_n(x) + \alpha_1 L^{-1} \sum_{n=0}^{\infty} A_n(x), \tag{23}$$

$$\sum_{n=0}^{\infty} v_n(x) = 1 - \alpha_2 L_1^{-1} \sum_{n=0}^{\infty} A_n(x) + \alpha_2 L^{-1} \sum_{n=0}^{\infty} A_n(x). \tag{24}$$

Next we establish the corresponding system of coupled Duan–Rach modified recursion schemes [15,16] as

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= (k-1)x - \alpha_1 x L_1^{-1} A_0(x) + \alpha_1 L^{-1} A_0(x), \\ u_{n+2}(x) &= -\alpha_1 x L_1^{-1} A_{n+1}(x) + \alpha_1 L^{-1} A_{n+1}(x), \quad n \geq 0, \end{aligned} \tag{25}$$

$$v_0(x) = 1,$$
$$v_{n+1}(x) = -\alpha_2 L_1^{-1} A_n(x) + \alpha_2 L^{-1} A_n(x), \quad n \geq 0, \tag{26}$$

from which we can readily calculate the respective second and third solution components as

$$u_1 = -\frac{x\left((1-x)\alpha_1 + 2(1-k)\left(1+\beta_1+\beta_2\right)\right)}{2\left(1+\beta_1+\beta_2\right)},$$
$$v_1 = -\frac{\left(1-x^2\right)\alpha_2}{2\left(1+\beta_1+\beta_2\right)},$$
$$u_2 = \frac{x\alpha_1}{24\left(1+\beta_1+\beta_2\right)^3}\left(5\alpha_2\left(1+\beta_1\right)+\alpha_1\left(1+\beta_2\right)+4(1-k)\left(1+\beta_2\right)\right.$$
$$\times\left(1+\beta_1+\beta_2\right) - x\left(\left(6-x^2\right)\alpha_2\left(1+\beta_1\right)+(2-x)x\alpha_1\left(1+\beta_2\right)\right.$$
$$\left.\left.+4(1-k)x\left(1+\beta_2\right)\left(1+\beta_1+\beta_2\right)\right)\right),$$
$$v_2 = \frac{\alpha_2}{24\left(1+\beta_1+\beta_2\right)^3}\left(5\alpha_2\left(1+\beta_1\right)+\alpha_1\left(1+\beta_2\right)+4(1-k)\left(1+\beta_2\right)\right.$$
$$\times\left(1+\beta_1+\beta_2\right) - x^2\left(\left(6-x^2\right)\alpha_2\left(1+\beta_1\right)+(2-x)x\alpha_1\left(1+\beta_2\right)\right.$$
$$\left.\left.+4(1-k)x\left(1+\beta_2\right)\left(1+\beta_1+\beta_2\right)\right)\right).$$

Thus we have effectively decomposed the constants of integration in (10) and (11) by formula. Furthermore, it becomes a straightforward procedure to automate calculation of additional solution components using an available computer algebra system such as MATHEMATICA, MAPLE or MATLAB, etc. by our approach since the analytic evaluation of all integrals in our subsequent solution components will be trivial, i.e. integrating powers of the independent variable x. In contrast, we checked that the analytic integration required for higher-order terms in [17], e.g. even $u_2(x)$ and $v_2(x)$, is quite time-consuming even using MATHEMATICA if the integrations include the parameters β_1 and β_2. The approximate solution functions as defined by Adomian and collaborators are

$$\phi_{m+1}(x) = \sum_{n=0}^{m} u_n(x), \quad \psi_{m+1}(x) = \sum_{n=0}^{m} v_n(x), \quad m \geq 0. \tag{27}$$

In order to examine their accuracy, we shall consider the appropriate error remainder functions

$$ER_{1,n}(x) = \frac{d^2}{dx^2}\phi_n(x) - \alpha_1 f\left(\phi_n(x), \psi_n(x)\right),$$
$$ER_{2,n}(x) = \frac{d^2}{dx^2}\psi_n(x) - \alpha_2 f\left(\phi_n(x), \psi_n(x)\right), \tag{28}$$

and maximal error remainder parameters

$$MER_{1,n} = \max_{0 \le x \le 1} \left| ER_{1,n}(x) \right|, \; MER_{2,n} = \max_{0 \le x \le 1} \left| ER_{2,n}(x) \right|, \tag{29}$$

whenever the solutions are unknown in advance.

We note by the Adomian–Rach modified recursion scheme, also known as the double decomposition method, in the Adomian decomposition method such as in [17], that the calculated solution components incorporate the undetermined coefficients, which are determined from successively matching the boundary conditions; see "Appendix 2".

By the Duan–Rach modified recursion scheme, we can easily calculate the solution components without any undetermined coefficients and with all of the modelling parameters. The results are shown to be superior for parametric simulations.

3 Numerical simulations

First, we assign $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = 3$ and $k = 0.5$, then calculate the error remainder functions, the maximal error remainder parameters and the approximate solutions.

The curves of the error remainder functions $ER_{1,n}(x)$ and $ER_{2,n}(x)$ versus x for $n = 3, 4, 5$ are plotted in Figs. 1 and 2, respectively, where the last curve overlaps nearly the x-axis.

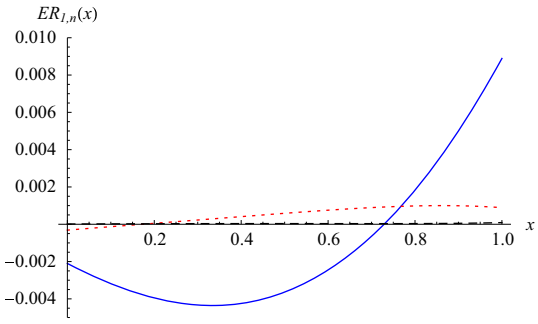

Fig. 1 Curves of the error remainder functions $ER_{1,n}(x)$ versus x for $n = 3$ (*solid line*), $n = 4$ (*dot line*) and $n = 5$ (*dash line*)

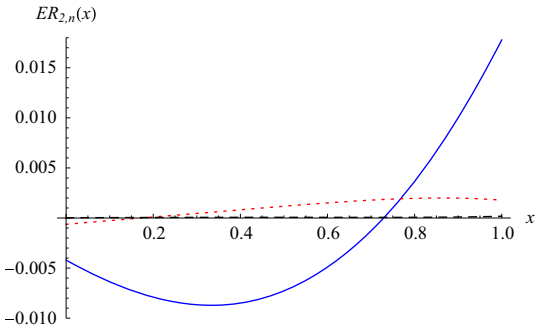

Fig. 2 Curves of the error remainder functions $ER_{2,n}(x)$ versus x for $n = 3$ (*solid line*), $n = 4$ (*dot line*) and $n = 5$ (*dash line*)

Table 1 The maximal error remainder parameters $MER_{1,n}$ and $MER_{2,n}$

n	$MER_{1,n}$	$MER_{2,n}$
1	0.2	0.4
2	0.0888889	0.177778
3	0.00888889	0.0177778
4	0.00099943	0.00199886
5	0.0000888889	0.000177778
6	8.88889×10^{-6}	0.0000177778
7	8.88889×10^{-7}	1.77778×10^{-6}
8	1.04915×10^{-7}	2.09831×10^{-7}
9	2.7602×10^{-8}	5.5204×10^{-8}
10	2.8038×10^{-9}	5.6076×10^{-9}

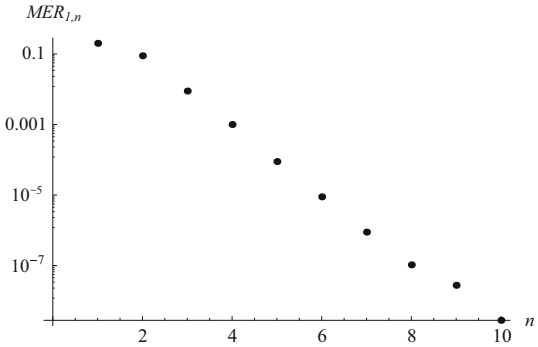

Fig. 3 Logarithmic plots of $MER_{1,n}$ versus n for $n = 1$ through 10

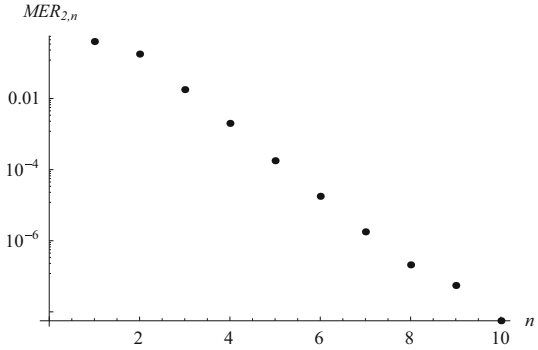

Fig. 4 Logarithmic plots of $MER_{2,n}$ versus n for $n = 1$ through 10

The maximal error remainder parameters $MER_{1,n}$ and $MER_{2,n}$, for $n = 1$ through 10, are listed in Table 1. The logarithmic plots of these values are displayed in Figs. 3 and 4, respectively, where the points almost lay on a straight line thus indicating an approximately exponential rate of convergence.

In Figs. 5 and 6, we plot the curves of the approximate solutions $\phi_n(x)$ and $\psi_n(x)$ versus x for $n = 2, 3, 4, 5$. In Fig. 5, the last three curves nearly overlap and in Fig. 6, the last two curves nearly overlap.

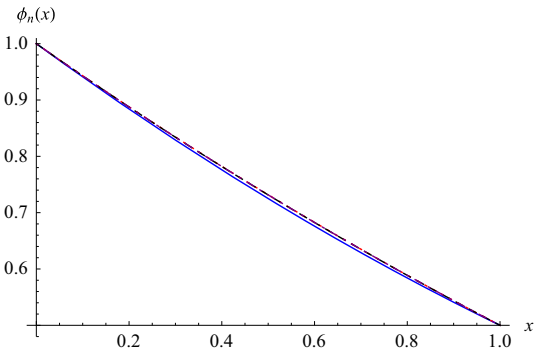

Fig. 5 Curves of the approximate solutions $\phi_n(x)$ versus x for $n = 2$ (*solid line*), $n = 3$ (*dot line*), $n = 4$ (*dash line*) and $n = 5$ (*dot-dash line*)

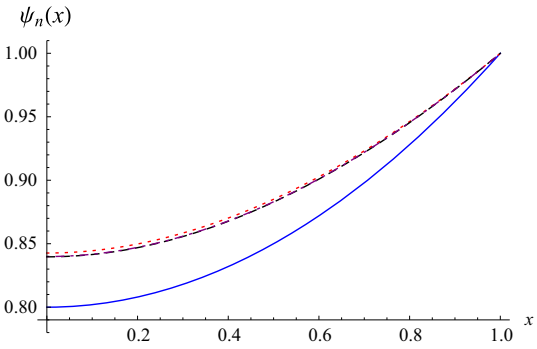

Fig. 6 Curves of the approximate solutions $\psi_n(x)$ versus x for $n = 2$ (*solid line*), $n = 3$ (*dot line*), $n = 4$ (*dash line*) and $n = 5$ (*dot-dash line*)

Next, we consider the enhancement factor of CO_2, which is defined in terms of the flux as

$$\eta = -\left.\frac{du}{dx}\right|_{x=0}. \tag{30}$$

We assign $\beta_1 = 1, \beta_2 = 3$ and $k = 0.5$, and then calculate the approximation $\phi_7(x; \alpha_1, \alpha_2)$ as parametrized by α_1 and α_2. Using $\phi_7(x; \alpha_1, \alpha_2)$, we obtain the enhancement factor η as a function of α_1 and α_2.

In Fig. 7, the surface of the enhancement factor $\eta(\alpha_1, \alpha_2)$ is displayed. In Fig. 8, the gradient field of the enhancement factor $\eta(\alpha_1, \alpha_2)$ is shown. It is evident that the enhancement factor η increases as α_1 increases, and decreases as α_2 increases.

Similarly, we assign $\alpha_1 = 1, \alpha_2 = 2$ and $k = 0.5$, and then calculate the approximation $\phi_7(x; \beta_1, \beta_2)$ as parametrized by β_1 and β_2. Furthermore, we obtain the enhancement factor η as a function of β_1 and β_2. In Fig. 9, the surface of the enhancement factor $\eta(\beta_1, \beta_2)$ is displayed. In Fig. 10, the gradient field of the enhancement factor $\eta(\beta_1, \beta_2)$ is shown. From these figures, we observe that the effect of the parameters β_1 and β_2 on the enhancement factor η rapidly weakens as β_1 or β_2 increases.

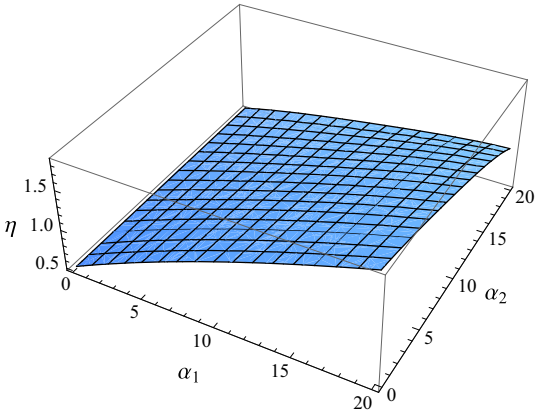

Fig. 7 The surface of the enhancement factor $\eta(\alpha_1, \alpha_2)$

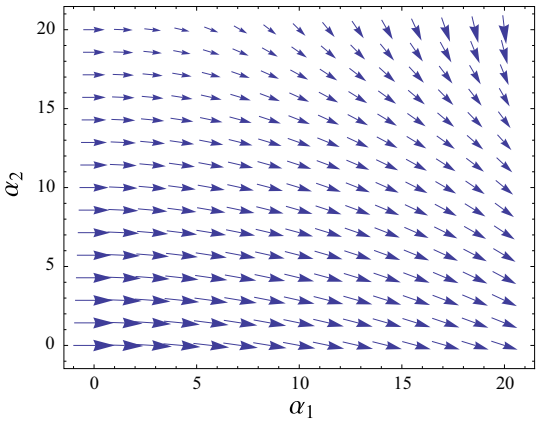

Fig. 8 The gradient field of the enhancement factor $\eta(\alpha_1, \alpha_2)$

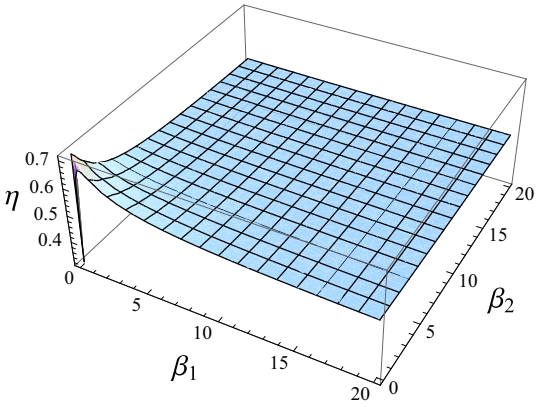

Fig. 9 The surface of the enhancement factor $\eta(\beta_1, \beta_2)$

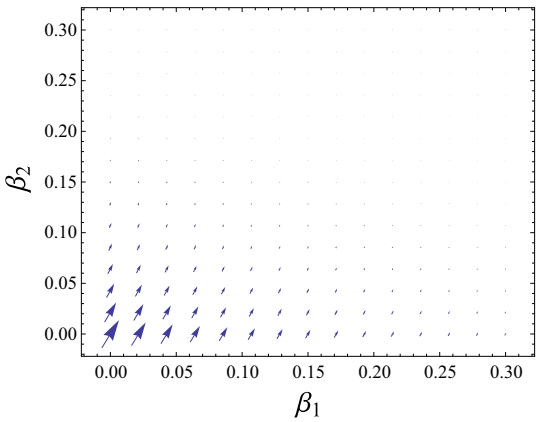

Fig. 10 The gradient field of the enhancement factor $\eta(\beta_1, \beta_2)$

4 Conclusions

In this work, we have examined a system of nonlinear differential equations, that relates the steady-state concentrations of carbon dioxide and PGE in solution, subject to the prescribed boundary conditions. The proposed approach depends mainly on combining the Adomian method with the Duan–Rach modified recursion scheme. Our analysis generated a rapidly convergent sequence of approximations of the concentrations of carbon dioxide and PGE to a high degree of accuracy. The evaluated approximations show enhancements over existing techniques where the minimal size of the obtained errors as well as the illustrated graphs emphasize these improvements.

Moreover, the enhancement factor of carbon dioxide was calculated for a variety of parameters. The Adomian decomposition method yields a rapidly convergent, easily computable, and readily verifiable sequence of analytic approximate solutions that is suitable for numerical parametric simulations. In closing, the proposed analysis that we applied is validated by its reliability and efficiency to analytically solve and numerically simulate various chemical engineering reactions.

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Appendix 1: MATHEMATICA code for the two-variable Adomian polynomials based on Theorem 1 [14]

```
Adth1[M_]:=Module[{},A[0]=f[Subscript[u, 0],Subscript[v, 0]];
For[n=1,n<=M,n++,A[n]=1/n*
Sum[(k+1)*(Subscript[u, k+1]*D[A[n-1-k],Subscript[u, 0]]
+Subscript[v, k+1]*D[A[n-1-k],Subscript[v, 0]]),{k,0,n-1}]];
Table[A[n],{n,0,M}]]
```

Appendix 2: The technique used in [17]

Consider the nonlinear differential equations in Eqs. (5) and (6),

$$Lu = \alpha_1 f\left(u\left(x\right), v\left(x\right)\right), \tag{31}$$
$$Lv = \alpha_2 f\left(u\left(x\right), v\left(x\right)\right), \tag{32}$$

where the linear differential operator L and the composite nonlinearity are

$$L\left(\cdot\right) = \frac{d^2}{dx^2}\left(\cdot\right), \quad f\left(u\left(x\right), v\left(x\right)\right) = \frac{u\left(x\right) v\left(x\right)}{1 + \beta_1 u\left(x\right) + \beta_2 v\left(x\right)}. \tag{33}$$

In the double decomposition method, the inverse linear operator L^{-1} is taken as a two-fold indefinite integration for second-order differential equations [7,8], i.e.

$$L^{-1}(\cdot) = C_0 + C_1 x + I_x^2(\cdot) = C_0 + C_1 x + \int\int(\cdot)\, dxdx, \tag{34}$$

where C_0 and C_1 are the constants of integration, which are called the matching coefficients, and where $I_x^2(\cdot) = \int\int(\cdot)\, dxdx$ denotes pure integrations. Applying the operator L^{-1} to both sides of Eqs. (31) and (32) yields the system of coupled nonlinear integral equations

$$u(x) = C_0 + C_1 x + \alpha_1 I_x^2 f\left(u\left(x\right), v\left(x\right)\right), \tag{35}$$
$$v(x) = D_0 + D_1 x + \alpha_2 I_x^2 f\left(u\left(x\right), v\left(x\right)\right), \tag{36}$$

where C_0, C_1, D_0, D_1 are arbitrary constants of integration to be determined by decomposition and matching at the boundaries for each stage of approximation.

The double decomposition method decomposes the solution $u(x)$, $v(x)$, the nonlinearity $f(u, v)$, and the matching coefficients C_i and D_i as

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \; v(x) = \sum_{n=0}^{\infty} v_n(x), \; f(u, v) = \sum_{n=0}^{\infty} A_n, \tag{37}$$
$$C_i = \sum_{n=0}^{\infty} C_{i,n}, \; D_i = \sum_{n=0}^{\infty} D_{i,n}, \quad i = 0, 1. \tag{38}$$

Upon substitution of these series into Eqs. (35) and (36), we can design the recursion scheme as

$$\begin{aligned} u_0 &= C_{0,0} + C_{1,0} x, \; v_0 = D_{0,0} + D_{1,0} x, \\ u_{n+1} &= C_{0,n+1} + C_{1,n+1} x + \alpha_1 I_x^2 A_n, \; n \geq 0, \\ v_{n+1} &= D_{0,n+1} + D_{1,n+1} x + \alpha_2 I_x^2 A_n, \; n \geq 0. \end{aligned} \tag{39}$$

Note that these necessary matching coefficients for u_{n+1} and v_{n+1} were omitted from recurrence relations in [17].

The constants $C_{i,n}$ and $D_{i,n}, i = 0, 1$, are determined by matching each of the partial sums $\phi_n(x)$ and $\psi_n(x)$, $n = 1, 2, \ldots$, to their respective boundary values [5,7,8] in Eqs. (3) and (4). This procedure can be carried out by matching u_0, v_0 to the given boundary values in Eqs. (3) and (4) to determine the values of $C_{i,0}$ and $D_{i,0}, i = 0, 1$, matching u_1, v_1 to the corresponding homogeneous boundary values $u_1(0) = u_1(1) = v_1'(0) = v_1(1) = 0$ to determine the values of $C_{i,1}$ and $D_{i,1}, i = 0, 1, \ldots$, and matching u_n, v_n to the corresponding homogeneous boundary values $u_n(0) = u_n(1) = v_n'(0) = v_n(1) = 0$ to determine the values of $C_{i,n}$ and $D_{i,n}, i = 0, 1$.

Matching u_0 and v_0 to the boundary values in Eqs. (3) and (4) determines

$$u_0 = 1 + (k-1)x, \;\; v_0 = 1.$$

Calculating u_1 and v_1 using (39) and matching u_1 and v_1 to the corresponding homogeneous boundary values $u_1(0) = u_1(1) = v_1'(0) = v_1(1) = 0$ determine that

$$\begin{aligned}
u_1 = {} & \frac{\alpha_1(x-1)x}{2\beta_1} + \frac{\alpha_1\,(\beta_2+1)\,x}{\beta_1^3(k-1)^2}\,(\beta_2+\beta_1 k+1)\log(\beta_2+\beta_1 k+1) \\
& - \frac{\alpha_1\,(\beta_2+1)}{\beta_1^3(k-1)^2}\,(\beta_1+\beta_2+\beta_1(k-1)x+1)\log(\beta_1+\beta_2+\beta_1(k-1)x+1) \\
& - \frac{\alpha_1\,(\beta_2+1)}{\beta_1^3(k-1)^2}(x-1)\,(\beta_1+\beta_2+1)\log(\beta_1+\beta_2+1)\,, \\
v_1 = {} & \frac{(x-1)\alpha_2}{2(k-1)\beta_1^2}\,((k-1)(1+x)\beta_1+2\,(1+\beta_2)) + \frac{(x-1)\alpha_2\,(1+\beta_2)}{(k-1)\beta_1^2} \\
& \times \log(\beta_1+\beta_2+1) + \frac{\alpha_2\,(1+\beta_2)}{(k-1)^2\beta_1^3}\,(\beta_2+\beta_1 k+1)\log(\beta_2+\beta_1 k+1) \\
& - \frac{\alpha_2\,(1+\beta_2)}{(k-1)^2\beta_1^3}\,(\beta_1+\beta_2+\beta_1(k-1)x+1)\log(\beta_1+\beta_2+\beta_1(k-1)x+1)\,.
\end{aligned}$$

Note that there are several errors in the expressions (B.13) and (B.15) as published in [17]. We have checked by using MATHEMATICA that it is quite time-consuming to calculate u_2 and v_2 and it is not at all feasible to calculate u_3 and v_3 if the results are parametrized by β_1 and β_2.

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